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On the Schrödinger equation for the interaction $x^2+\lambda x^2/(1+gx^2)$

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Abstract. The method of obtaining the explicit expressions of all the even- and odd-parity exact solutions of the Schrödinger equation for the interaction $x^2+\lambda x^2/(1+gx^2)$ is discussed when the couplings λ and g satisfy some specific relations. In the general case a simple equation for approximating the energy eigenvalues has been developed.

1. Introduction

The importance of the one-dimensional non-polynomial interaction Lagrangian of the kind $x^2+\lambda x^2/(1+gx^2)$ has been pointed out by a number of authors in connection with the nonlinear Lagrangian field theory (Risken and Vollmer 1967) and nonlinear optics (Haken 1970). The Schrödinger equation with such an interaction Lagrangian is the analogue of a zero-dimensional field theory with a nonlinear Lagrangian which is used in elementary particle physics (Biswas *et al* 1973, Salam and Strathdee 1970).

The Schrödinger equation

$$(\mathrm{d}^2/\mathrm{d}x^2+E-V(x))\psi(x)=0 \tag{1}$$

with the potential

$$V(x)=x^2+\lambda x^2/(1+gx^2) \qquad -\infty<x<\infty \tag{2}$$

has recently been studied by many authors using different variational techniques (Mitra 1978, Bessis and Bessis 1980), the Padé approximant method (Lai and Lin 1982), the Hill determinant method (Hautot 1981), the asymptotic series expansion scheme (Kaushal 1979), the finite difference method (Galicia and Killingbeck 1979) and the perturbed operators method (Bessis *et al* 1983). The existence of a class of exact solutions for particular values of λ and g has recently been shown by Flessas (1981), Varma (1981), Whitehead *et al* (1982) and Znojil (1983). Whitehead *et al* (1982) have also presented a number of theorems regarding the general nature of these solutions. Znojil (1983) has proceeded along the same lines as Whitehead *et al* (1982) and has constructed the exact wavefunctions and an analytic continued-fractional Green function for the Schrödinger equation (1) with the potential (2) and with a more general potential.

In the present paper we would like to make a systematic study of the exact even- and odd-parity solutions in the form of products of exponential and polynomial

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functions of x. For the existence of these types of solutions it is necessary that λ and g should be related. Our method described in § 2 is suitable for numerical evaluation of all the coefficients of the polynomials of x and the corresponding connection between λ and g which is identical to that given by Whitehead *et al* (1982). In § 3 we develop an approximation scheme which is simple and accurate for obtaining the energy eigenvalues in the general case. The perturbation calculation for the interaction (2) is made by expanding the factor $1/(1+gx^2)$ in a power series for gx^2. Lai and Lin (1982) have applied the Hellmann–Feynman theorem and hypervirial theorems to the perturbation series to calculate the energy eigenvalues. They have also employed the hypervirial relations (Hirschfelder 1960, Swenson and Danforth 1972) and the Padé approximant method (Baker 1965, Loeffel *et al* 1969, Killingbeck 1978, Lai 1981) to the energy series. The results, however, require the asymptotic expansion of the factor $1/(1+gx^2)$ in a power series of gx^2 which is valid for low values of $g(\leqslant 2)$ only. The variational calculations on the other hand require very elaborate numerical computations. We would like to solve the problem in a completely different way. We replace a slowly varying function by a constant and obtain a simple expression for the energy eigenvalues. The results of the present calculation are in good agreement with the existing results.

2. Exact solutions to the Schrödinger equation

First of all we make the standard substitution

$$\psi(x)=\exp(-x^2/2)\phi(x) \tag{3}$$

to transform the Schrödinger equation into the form

$$(1+gx^2)[\phi''(x)-2x\phi'(x)]+[E-1+x^2(Eg-g-\lambda)]\phi(x)=0. \tag{4}$$

It is clear from this equation that $x=0$ is an ordinary point and $x=\infty$ is an irregular singular point of the differential equation when $g>0$. Therefore equation (4) admits a convergent series solution about $x=0$. The infinite series must be truncated in order to satisfy the boundary conditions

$$\lim_{x\to\pm\infty}\psi(x)=0 \tag{5}$$

for the normalisation of the wavefunction. Thus we may assume the even- and odd-parity solutions of equation (4) in the form of polynomials of x

$$\phi^{\mathrm{e}}_{2n}(x)=1+\sum_{m=1}^{n}a_{2m}x^{2m} \tag{6}$$

$$\phi^{\mathrm{o}}_{2n+1}(x)=x+\sum_{m=1}^{n}a_{2m+1}x^{2m+1} \tag{7}$$

where the overall normalisation constants of the wavefunctions have been omitted. The superscripts 'e' and 'o' in equations (6) and (7) refer to the even- and odd-parity solutions.

We substitute (6) and (7) into (4) to obtain the recurrence relations between the coefficients $A_{2n,2n+1}$. These relations may be written in matrix notation:

$$\begin{pmatrix} b_{11} & 0 & 0 & & \cdots \\ b_{21} & b_{22} & 0 & & \cdots \\ b_{31} & b_{32} & b_{33} & & \\ 0 & b_{42} & b_{43} & \ddots & \\ \vdots & \vdots & \vdots & & b_{nn} \end{pmatrix} \begin{pmatrix} a_{2+\nu} \\ a_{4+\nu} \\ \\ \vdots \\ a_{2n+\nu} \end{pmatrix} = \begin{pmatrix} -4n-\lambda' \\ -4ng \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{8}$$

with

$$[(2n-1+\nu)(2n+\nu)g+\lambda']a_{2n+\nu}+4ga_{2n-2+\nu}=0 \qquad n\geqslant 2 \tag{9}$$

and

$$E_n^{\mathrm{e,o}}=4n+1+\lambda'+2\nu \qquad n\geqslant 0 \tag{10}$$

where $\nu=0$ for the even-parity solutions and $\nu=1$ for the odd-parity solutions, $\lambda'=\lambda/g$, and the non-zero matrix elements b_{ij} of the equation (8) are given by

$$b_{ii}=(2i-1+\nu)(2i+\nu) \tag{11a}$$

$$b_{ii-1}=(2i-3+\nu)(2i-2+\nu)g+4(n-i+1)+\lambda' \tag{11b}$$

$$b_{ii-2}=4(n-i+2)g \qquad i=1,2,\ldots,n. \tag{11c}$$

By solving (8) we can find all the coefficients in terms of λ and g and then equation (9) gives the relation between λ and g. Equation (8) has a unique solution because the determinant of the square matrix b_{ij} is $(2n+\nu)!$. The first four eigenvalues, the unnormalised eigenfunctions (apart from the exponential factor) and the relations between λ' and g are given below:

(1) $\phi_0^{\mathrm{e}}(x)=1 \qquad E_0^{\mathrm{e}}=1 \qquad \lambda'=0$

(2) $\phi_1^{\mathrm{o}}(x)=x \qquad E_1^{\mathrm{o}}=3 \qquad \lambda'=0$

(3) $\phi_2^{\mathrm{e}}(x)=1-\frac{1}{2}(4+\lambda')x^2 \qquad E_2^{\mathrm{e}}=5+\lambda'$

$\lambda'(\lambda'+2g+4)=0$

(4) $\phi_3^{\mathrm{o}}(x)=x-\frac{1}{6}(4+\lambda')x^3 \qquad E_3^{\mathrm{o}}=7+\lambda'$

$\lambda'(\lambda'+6g+4)=0.$

For any value of n when the wavefunctions are given by (6) and (7) the coefficients may be determined from (8) by the application of Cramer's rule. For example $a_{2n-2+\nu}$ and $a_{2n+\nu}$ are given by

$$a_{2n-2+\nu}=\frac{1}{(2n+\nu)!}\begin{vmatrix} b_{11} & 0 & \cdots & -4n-\lambda' & 0 \\ b_{21} & b_{22} & \cdots & -4ng & 0 \\ b_{31} & b_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & 0 & b_{nn} \end{vmatrix}$$

$$a_{2n+\nu} = \frac{1}{(2n+\nu)!} \begin{vmatrix} b_{11} & 0 & \dots & 0 & -4n-\lambda' \\ b_{21} & b_{22} & \dots & 0 & -4ng \\ b_{31} & b_{32} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn-1} & 0 \end{vmatrix} .$$

The condition for the existence of the solutions is given by (9). A little manipulation will show that this condition is identical to that obtained by Whitehead *et al* (1982, equation (2.14)) and therefore following Whitehead *et al* (1982) we may state the following important properties regarding the nature of solutions of equation (9): (i) $\lambda' = 0$ is always one of the roots, (ii) all the roots are real and (iii) the non-zero roots are all negative.

Here we have given explicit expressions of all the even- and odd-parity solutions. Equations (8) and (9) are suitable for numerical evaluation of all the eigenfunctions for specific values of λ' and g given by (9).

3. Approximate eigenvalues

In the general case when $g > 0$ we develop a method of obtaining the approximate eigenvalues of the Schrödinger equation (1) with the interaction (2). We may write equation (1) in two forms:

$$(\mathrm{d}^2/\mathrm{d}x^2 + E - x^2 - \lambda' f_1(x))\psi(x) = 0 \tag{12}$$

$$(\mathrm{d}^2/\mathrm{d}x^2 + E - x^2 - \lambda' + \lambda' f_2(x))\psi(x) = 0 \tag{13}$$

where

$$f_1(x) = gx^2/(1+gx^2) \tag{14}$$

$$f_2(x) = 1/(1+gx^2). \tag{15}$$

As x varies from $-\infty$ to $+\infty$ the function $f_1(x)$ runs from 1 to 1 through zero at $x = 0$, whereas $f_2(x)$ runs from 0 to 0 through 1 at $x = 0$, $f_1(x)$ and $f_2(x)$ being non-negative always. If the variations of these two functions are assumed to be small and they are replaced by their expectation values $\langle f_1 \rangle$ and $\langle f_2 \rangle$ in the considered quantum state, equations (12) and (13) become harmonic oscillator equations. We take the average of the two eigenvalues thus obtained, since it may be assumed that the error made by replacing $f_1(x)$ by $\langle f_1 \rangle$ will be compensated by the error made by replacing $f_2(x)$ by $\langle f_2 \rangle$. Both the functions $f_1(x)$ and $f_2(x)$ vary between 0 and 1 and the minimum of $f_1(x)$ falls at the maximum of $f_2(x)$ and *vice versa*, and moreover $f_1(x)$ is concave upwards whereas $f_2(x)$ is concave downwards. So the errors go in the opposite directions in the two cases and will cancel each other when added provided λ' is small. According to this scheme the energy eigenvalues are given by

$$E_n = 2n + 1 + \tfrac{1}{2}\lambda' - \lambda'(\sqrt{\pi}\, 2^n n!)^{-1} I_n \tag{16}$$

where

$$I_n = \int_0^{\infty} \exp(-x^2) H_n^2(x)(1-gx^2)/(1+gx^2)\, \mathrm{d}x \tag{17}$$

with $n = 0, 1, 2, \dots$ and $H_n(x)$ a Hermite polynomial of order n. The integrations of equation (17) can be performed easily by using the following results (Gradshteyn and Ryzhik 1965):

$$\int_0^\infty \exp(-x^2)x^{2n}/(1+gx^2)\,dx$$
$$=\tfrac{1}{2}\exp(1/g)g^{-n-1/2}\Gamma(n+\tfrac{1}{2})\Gamma(\tfrac{1}{2}-n, 1/g) \qquad n\geqslant 0 \tag{18}$$

where $\Gamma(a, x)$ is the incomplete gamma function having the series expansion

$$\Gamma(a, x)=\Gamma(a)-\sum_{n=0}^{\infty}\frac{(-1)^n x^{a+n}}{n!(a+n)}. \tag{19}$$

The results of the first four integrals I_n for any positive value of g are given below:

$$I_0=\tfrac{1}{2}g^{-1/2}\exp(1/g)[\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}, 1/g)-\Gamma(\tfrac{3}{2})\Gamma(-\tfrac{1}{2}, 1/g)] \tag{20a}$$

$$I_1=2g^{-3/2}\exp(1/g)[\Gamma(\tfrac{3}{2})\Gamma(-\tfrac{1}{2}, 1/g)-\Gamma(\tfrac{5}{2})\Gamma(-\tfrac{3}{2}, 1/g)] \tag{20b}$$

$$I_2=\exp(1/g)(-8g^{-5/2}\Gamma(\tfrac{7}{2})\Gamma(-\tfrac{5}{2}, 1/g)+8(1+g)g^{-5/2}\Gamma(\tfrac{5}{2})\Gamma(-\tfrac{3}{2}, 1/g)$$
$$-(8+2g)g^{-3/2}\Gamma(\tfrac{3}{2})\Gamma(-\tfrac{1}{2}, 1/g)+2g^{-1/2}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}, 1/g)) \tag{20c}$$

$$I_3=\exp(1/g)(-32g^{-7/2}\Gamma(\tfrac{9}{2})\Gamma(-\tfrac{7}{2}, 1/g)+(32+96g)g^{-7/2}\Gamma(\tfrac{7}{2})\Gamma(-\tfrac{5}{2}, 1/g)$$
$$-(96+72g)g^{-5/2}\Gamma(\tfrac{5}{2})\Gamma(-\tfrac{3}{2}, 1/g)+72g^{-3/2}\Gamma(\tfrac{3}{2})\Gamma(-\tfrac{1}{2}, 1/g)). \tag{20d}$$

With the help of these equations and (19) the first four eigenvalues can be calculated easily. They are given in table 1 for $g=0.5$, 1, 2, 5, 10, 20, 100 and 500. We find that for small values of λ', the eigenvalues are linear functions of λ'. The equations can also be used to evaluate the energy eigenvalues for small negative values of λ'. The first four eigenvalues for $g\leqslant 2$ obtained by this method are compared with the existing results in table 2. The agreement in general is excellent. When g is large we have to take only a few terms of series (19) of the incomplete gamma function and the calculation becomes simple. The expressions of the first four eigenvalues when g is sufficiently large are given below:

$$E_0=1+\lambda'(1-\sqrt{\pi}\,g^{-1/2}+\tfrac{5}{2}g^{-1}) \tag{21a}$$

$$E_1=3+\lambda'(1-\tfrac{3}{2}g^{-1}+2\sqrt{\pi}\,g^{-3/2}) \tag{21b}$$

$$E_2=5+\lambda'(1-\tfrac{1}{2}\sqrt{\pi}\,g^{-1/2}+\tfrac{9}{4}g^{-1}) \tag{21c}$$

$$E_3=7+\lambda'(1-\tfrac{3}{2}g^{-1}+\tfrac{3}{2}\sqrt{\pi}\,g^{-3/2}). \tag{21d}$$

Table 1. The expressions of the first four eigenvalues for $g=0.5$, 1, 2, 5, 10, 20, 100, 500 and small values of λ.

g	E_0-1	E_1-3	E_2-5	E_3-7
0.5	0.314 5246λ	0.741 903λ	0.931 4597λ	0.997 3218λ
1	0.242 1296λ	0.515 74λ	0.589 5768λ	0.648 9344λ
2	0.172 1604λ	0.327 8394λ	0.344 3208λ	0.374 2396λ
5	0.097 9383λ	0.160 8245λ	0.155 9797λ	0.169 855λ
10	0.059 4434λ	0.088 1112λ	0.082 7987λ	0.090 3768λ
20	0.034 3377λ	0.046 5662λ	0.043 273λ	0.047 105λ
100	0.008 411λ	0.009 8317λ	0.009 2574λ	0.009 8449λ
500	0.001 8451λ	0.001 9926λ	0.001 9279λ	0.001 9929λ

Table 2. The first four eigenvalues (correct up to five decimal places) obtained by four different methods for $g = 0.5$, 1 and 2.

	Mitra (1978)	Bessis and Bessis (1980)	Lai and Lin (1982)	The present calculation
	1.031 21	1.031 21	1.031 21	1.031 45
$\lambda = 0.1$	3.073 89	3.073 90	3.073 90	3.074 19
$g = 0.5$	5.093 05	5.093 07	5.093 06	5.093 15
	—	7.105 85	7.105 84	7.099 73
	1.151 56	1.151 56	1.151 56	1.157 26
$\lambda = 0.5$	3.363 80	3.363 80	3.363 80	3.370 95
$g = 0.5$	5.463 20	5.463 21	5.463 21	5.465 73
	—	7.527 88	7.527 89	7.498 66
	1.292 95	1.292 95	1.292 95	1.314 52
$\lambda = 1$	3.713 90	3.713 90	3.713 90	3.741 90
$g = 0.5$	5.920 63	5.920 63	5.920 63	5.931 46
	—	8.052 38	8.052 44	7.997 32
	1.024 10	1.024 19	1.024 12	1.024 21
$\lambda = 0.1$	3.051 49	3.051 65	3.051 53	3.051 57
$g = 1$	5.034 44	5.059 29	5.058 99	5.058 96
	—	7.065 50	7.064 97	7.064 89
	1.118 54	1.118 59	1.118 55	1.121 06
$\lambda = 0.5$	3.255 77	3.255 84	3.255 80	3.257 87
$g = 1$	5.294 88	5.295 06	5.294 92	5.294 79
	—	7.324 54	7.324 46	7.324 47
	1.232 35	1.232 37	1.232 35	1.242 13
$\lambda = 1$	3.507 38	3.507 42	3.507 40	3.515 74
$g = 1$	5.589 77	5.589 86	5.589 83	5.589 58
	—	7.648 32	7.649 07	7.648 93
	1.017 18	1.017 89	1.017 28	1.017 22
$\lambda = 0.1$	3.032 76	3.031 77	3.032 96	3.032 78
$g = 2$	5.034 44	5.055 85	3.034 55	5.034 43
	—	7.034 74	7.037 76	7.037 42
	1.085 19	1.087 06	1.085 29	1.086 08
$\lambda = 0.5$	3.163 46	3.186 78	3.163 71	3.163 92
$g = 2$	5.172 40	5.176 89	5.172 58	5.172 16
	—	7.226 54	7.188 01	7.187 15
	1.168 67	1.170 49	1.168 72	1.172 16
$\lambda = 1$	3.326 02	3.329 04	3.326 14	3.327 80
$g = 2$	5.345 24	5.348 49	5.345 64	5.344 32
	—	7.381 14	7.378 34	7.374 24

It is interesting to compare our results given by formula (16) and the asymptotic expressions (21) with the exact values of Bessis and Bessis (1980) for large values of g. The first four energy levels for $g = 100$ and 500 are given in table 3. From the tables it is clear that this simple scheme is suitable for excited state energy eigenvalues and for large values of g. As is expected, the eigenvalues decrease steadily with

Table 3. The first four energy levels for high values of g: (a) formula (16), (b) formula (21), (c) Bessis and Bessis (1980).

	The present calculation (a)	The expressions (21) (b)	Bessis and Bessis (1980) (c)
	1.000 841 1	1.000 847 8	1.000 841 1
$\lambda=0.1$	3.000 983 17	3.000 988 5	3.000 983 1
$g=100$	5.000 927 54	5.000 933 9	5.000 925 7
	7.000 984 49	7.000 987 7	7.000 984 5
	1.084 11	1.084 78	1.084 064 3
$\lambda=10$	3.098 317	3.098 85	3.098 317 0
$g=100$	5.092 754	5.093 39	5.092 762 46
	7.098 449	7.098 77	7.098 449 1
	1.841 1	1.847 75	1.836 385 0
$\lambda=100$	3.983 17	3.988 54	3.983 099 2
$g=100$	5.927 54	5.933 88	5.928 352 5
	7.984 49	7.987 66	7.984 444 8
	1.000 184 51	1.000 185 1	1.000 118 49
$\lambda=0.1$	3.000 199 26	3.000 199 5	3.000 199 2
$g=500$	5.000 192 79	5.000 193 0	5.000 192 8
	7.000 199 29	7.000 199 4	7.000 199 2
	1.184 51	1.185 15	1.184 863 2
$\lambda=100$	3.199 26	3.199 46	3.199 260 1
$g=500$	5.192 76	5.192 97	5.192 804 3
	7.199 29	7.199 45	7.199 287 9
	1.922 55	1.925 73	1.923 226 0
$\lambda=500$	3.996 30	3.997 32	3.996 296 9
$g=500$	5.963 95	5.964 87	5.964 116 1
	7.996 45	7.997 24	7.996 436 7

increasing g and fixed λ and approach the eigenvalues $2n+1$ ($n=0, 1, 2, \ldots$) of the harmonic oscillator asymptotically for large g.

4. Conclusion

In this paper we have presented a method of obtaining all the exact even- and odd-parity eigenvalues and eigenfunctions when λ and g are related by some specific relations. The eigenfunctions and the eigenvalues reduce to the harmonic oscillator eigenfunctions and eigenvalues as $\lambda \to 0$. In the general case a simple equation for approximating the energy eigenvalues has been developed. For large values of g and small λ/g it seems that the present method works well, whereas the asymptotic series expansion scheme (Kaushal 1979), Padé approximant method (Lai and Lin 1982) and the variational calculation of Mitra (1978) are restricted to low values of $g \leqslant 2$. The present scheme is simple and may be used for obtaining excited state energy eigenvalues. It is clear from (16) and (17) that we have in fact assumed that $(1-gx^2)/(1-$

gx^2) is a slowly varying function and we have taken its expectation value in the considered quantum state. It should be mentioned here that the method of approximating a slowly varying function by a constant has been used for a long time in the literature for the case of the Yukawa potential (Ecker and Weizel 1956, Lam and Varshni 1976, Talukdar *et al* 1978, Das *et al* 1979) and exponential cosine screened Coulomb potential (Dutt 1979, Ray and Ray 1980).

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